Problem No. 1 (Calculus)

Answer the following questions. e is the base of the natural logarithm and i is the imaginary unit.
Omit the derivations and write only the answers.

(Q.1) Define the real function $f(x)$ as follows:

$$f(x) = \int_0^x \mathrm{d}t\,\mathrm{e}^{-t^2}.$$

(1) Obtain the derivative of $f(x)$.

(2) Obtain the Taylor expansion of $f(x)$ around $x = 0$ up to the third order.

(Q.2) Two real functions $x(t)$ and $y(t)$ satisfy the following coupled differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\nu x - y + A\cos at,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\nu y + x + A\sin at,$$

where A, a, and ν ($\nu > 0$) are real constants.

(1) Let $z(t) = x(t) + \mathrm{i}y(t)$. Obtain the differential equation for $z(t)$.

(2) Let $A = 0$. Obtain the general solution $x(t)$ and $y(t)$ of the coupled differential equations.

(3) Let $A > 0$. $x(t)$ asymptotically approaches $x_1(t) = B\cos(bt + \phi)$ as $t \to \infty$ regardless of the initial condition. Here, B ($B > 0$), b, and ϕ are real constants. Obtain B, b, and ϕ.

(Q.3) Define the real function $f(x)$ as follows:

$$f(x) = x\mathrm{e}^{-x}.$$

(1) Obtain all the extrema of $f(x)$ and the corresponding values of x.

(2) In an xy Cartesian coordinate system, obtain the area of the region defined by $0 \leq y \leq f(x)$ and $x > 0$.

(3) In an xyz Cartesian coordinate system, obtain the volume of the solid formed by rotating the xy-plane region defined in (2) around the x-axis.

(4) The surface of the solid defined in (3) can be expressed in terms of the real function

$$g(x, y, z) = y^2 + z^2 - x^2\mathrm{e}^{-2x},$$

as $g(x, y, z) = 0$. Obtain the gradient vector of $g(x, y, z)$.

(5) Obtain the coordinates (x, y, z) where $h(x, y, z) = xyz$ is maximized on the surface of the solid defined in (3) in the $x > 0$ region. In addition, obtain the maximum.

Problem No. 2.1 (Linear algebra)

The transpose of a matrix or a vector is denoted by the superscript $\top$. The exponential of a square matrix J is defined as

$$\mathrm{e}^J = \sum_{k=0}^{\infty} \frac{1}{k!} J^k.$$

Answer the following questions.

(Q.1) Consider the following real matrix A and real vectors $\boldsymbol{b}$ and $\boldsymbol{c}$:

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ \beta \end{pmatrix}, \quad \boldsymbol{c} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix},$$

where β is a constant.

Omit the derivations and write only the answers.

(1) Obtain the eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3$ of A.

(2) Obtain the matrix P that satisfies

$$P^\top AP = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

(3) Suppose that the equation $A\boldsymbol{x} = \boldsymbol{b}$ for the variable $\boldsymbol{x} \in \mathbb{R}^3$ has a solution. Obtain the value of β. In addition, obtain all the solutions of the equation.

(4) Obtain $\lim_{t\to\infty} \mathrm{e}^{-At}\boldsymbol{c}$, where t is a real number.

(Q.2) Let B be an $n \times n$ real asymmetric matrix.

(1) Show that the eigenvalues of B and $B^\top$ are the same. You may use the fact that the determinants of a square matrix and its transposed matrix are the same.

(2) Let λ_i $(i = 1, \ldots, n)$ be the eigenvalues of B. Let $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$ be the eigenvectors of B and $B^\top$ corresponding to the eigenvalue λ_i, respectively. Namely,

$$B\boldsymbol{u}_i = \lambda_i \boldsymbol{u}_i,$$
$$B^\top \boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i$$

hold true. Show that $\boldsymbol{v}_j^\top \boldsymbol{u}_i = 0$ for $\lambda_i \neq \lambda_j$.

Problem No. 2.2 (Probability and Statistics)

Answer the following questions. e is the base of the natural logarithm, and i is the imaginary unit.
Omit the derivations and write only the answers.

(Q.1) Let X and Y be random variables with the means $\overline{X} = 1$, $\overline{Y} = 2$, the variances $s_X = 1$, $s_Y = 2$, respectively, and the covariance $s_{XY} = -1$. Random variables U and V are defined by $U = X + 3$ and $V = X + Y$.

(1) Obtain the means $\overline{U}$, $\overline{V}$ of U, V.

(2) Obtain the variances s_U, s_V of U, V.

(3) Obtain the covariance s_{UV} between U and V.

(Q.2) Let X be a random variable obeying the standard Cauchy distribution:

$$p(x) = \frac{1}{\pi\left(1 + x^2\right)}.$$

Answer the following questions.

(1) Obtain the cumulative distribution function $F(x)$ of the standard Cauchy distribution.

(2) Let U be a random variable obeying the uniform distribution on the interval $[0, 1]$. Obtain the probability $\Pr(U \leq F(x))$, where $F(x)$ is the cumulative distribution function in (1).

(3) For U defined in (2), obtain the real function $g(U)$ that satisfies $X = g(U)$.

(4) Obtain the characteristic function of the standard Cauchy distribution: $\psi(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx} p(x)\mathrm{d}x$, where t is a real number.

(5) Let $X_1, X_2, \ldots, X_n$ be n random variables that are independent and identically distributed and obey the standard Cauchy distribution. A random variable Z is defined by $Z = X_1 + X_2 + \cdots + X_n$. Obtain the probability density function of Z.

(Q.3) Let $(x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)$ be N sample pairs of random variables X and Y. Define the following statistics on the sample pairs:

$$\overline{x} = \frac{1}{N}\sum_{i=1}^{N} x_i, \quad \overline{y} = \frac{1}{N}\sum_{i=1}^{N} y_i,$$

$$\sigma_{xx} = \frac{1}{N}\sum_{i=1}^{N} x_i x_i, \quad \sigma_{xy} = \frac{1}{N}\sum_{i=1}^{N} x_i y_i.$$

The regression line $Y = A_0 + A_1 X$ is obtained by minimizing the following function l:

$$l(A_0, A_1) = \sum_{i=1}^{N} \left(y_i - A_0 - A_1 x_i\right)^2,$$

which is assumed to be minimum at $A_0 = a_0$ and $A_1 = a_1$. Express a_0 and a_1 in terms of $\overline{x}$, $\overline{y}$, σ_{xx}, and σ_{xy}. In addition, show the condition under which both a_0 and a_1 are uniquely determined.

Problem No. 2.3 (Mechanics)

Answer the following questions. Let the gravitational acceleration be $g\,(>0$, constant). Write only the answers.

(Q.1) A liner molecule ABA is composed of Atoms A and B. Atom B is located between two Atoms A. Consider Molecule ABA as a system composed of three point masses connected by two massless springs of spring constant k. The masses of Atoms A and B are m and M, respectively. The atoms move along a common line. Ignore the gravity.

(1) Write the equations of motion for the three atoms. The displacements of Atoms A, B, A from the natural lengths are defined as x, y, and z, respectively, with the positive direction from one Atom A to the other Atom A.

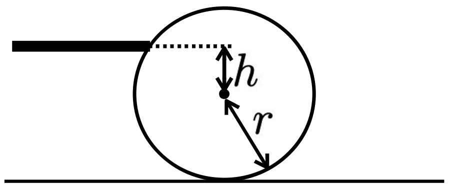
(2) Introduce $Q_1 = x + z$ and $Q_2 = x - z$. Answer the angular frequencies ω_1 and ω_2 of their oscillation. Let the center of the gravity of Molecule ABA do not move.

(Q.2) Consider a point mass thrown at an elevation angle θ with initial velocity $v\,(\neq 0)$. x axis is taken in the positive direction of the horizontal component of the initial velocity. y axis is taken in the vertical upward direction. The origin is the initial position of the point mass.

(1) Answer the vertical position y of the point mass at $x = X\,(> 0)$, using X, g, θ, and v.

(2) Answer the range of the vertical position y where the point mass cannot reach at $x = X\,(> 0)$ for any angles θ, using X, g, and v.

(Q.3) A uniform-density rigid sphere of mass m and radius r is at rest on a frictionless horizonal plate. We want to roll the sphere without slipping by hitting horizontally with a stick as shown in the figure below. Answer the vertical distance h from the center of the sphere to hit using m, r, and the moment of inertia I of the rigid sphere.



(Q.4) Suppose the Earth is a rigid sphere that rotates eastward around the axis passing through the south pole and the north pole at a constant angular velocity ω. Let (x, y, z) be the coordinate system fixed to the Earth's surface with the origin at Point P on the northern hemisphere at latitude $\theta\,(0 < \theta < \frac{\pi}{2})$. Positive direction of z axis is defined in the direction from the Earth's center to Point P, and positive directions of x axis and y axis are defined towards the south and east, respectively, on the tangential plane at Point P. A point mass of mass m is thrown from Point P to the positive direction of z axis at initial velocity v.

(1) Answer the x, y, and z components of the Earth's angular velocity vector $\boldsymbol{\omega}$ in the coordinate system (x, y, z).

(2) The equation of motion for the point mass in the coordinate system (x, y, z) is expressed as follows using position vector $\boldsymbol{r}$, its first derivative with respect to time $\dot{\boldsymbol{r}}$, and its second derivative $\ddot{\boldsymbol{r}}$:

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F} + 2m\dot{\boldsymbol{r}} \times \boldsymbol{\omega}$$

$$\boldsymbol{F} = (0, 0, -mg)$$

Write the acceleration for the point mass in the y direction at the time t elapsed after throwing the point mass, using, g, t, v, ω, θ. Ignore the terms of order ω^2 or higher.

Problem No. 2.4 (Electromagnetism)

Answer the following questions, assuming vacuum environment. Except for Q.5(2), write only the answers.

(Q.1) In the below, ε_0 and μ_0 are dielectric constant and permeability, respectively, in vacuum.

(1) Write the light speed, using ε_0 and μ_0 .

(2) Two conducting wires are placed in parallel at a distance r. The length of the wires is infinite. Current I is flowing in the same direction in each wire. Write the magnitude and direction of the force acting on the wire per unit length.

(Q.2) Answer the questions on the electric circuit shown in Figure 1. The R_1, R_2, and R_3 represent electric resistors, whose resistance values are r_1, r_2, and r_3, respectively.

(1) Write the combined resistance value between A and B, using $r_1,\ r_2,\ r_3$.

(2) A constant voltage V is applied between A and B. Write the current flowing in R_3, and write the electric power consumed in R_2. The answers should use $V,\ r_1,\ r_2,\ r_3$.

(Q.3) Answer the questions on the electric circuit shown in Figure 2. An AC voltage is applied with an amplitude $\tilde{V}$ and an angular frequency

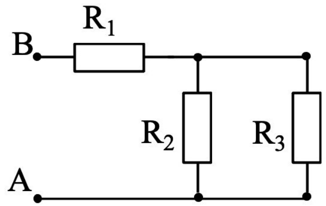


Figure 1

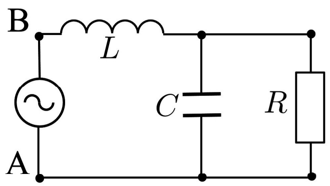


Figure 2

ω. Here, L is the inductance of the coil, C is the capacitance of the capacitor, R is the resistance value of the resistor. Use j as the imaginary unit.

(1) Write the combined impedance between A and B, using $\omega,\ L,\ C,\ R$.

(2) Write the ratio $\tilde{I}_C/\tilde{I}_R$, where $\tilde{I}_C$ is the amplitude of the current flowing in the capacitor and $\tilde{I}_R$ is the amplitude of the current flowing in the resistor.

(3) We remove the capacitor from the circuit, namely we handle a circuit without the capacitor. We express the voltage variation between A and B as $V = \tilde{V}e^{j\omega t}$. Write the current flowing in the coil, using $\omega,\ L,\ R,\ V$. Write also the effective electric power consumed in the resistor, using $\omega,\ L,\ R,\ \tilde{V}$. (The effective electric power is the time-averaged electric power.)

(Q.4) There is a closed circuit shown in Figure 3. The circuit is formed by one turn loop of a conducting wire and its surface area is S.

Using an equation

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

from Maxwell's equations, derive the expression on a voltage V in the circuit induced by the magnetic field crossing the circuit (Note: write only the expression on V). Here, $\boldsymbol{B}$ is a magnetic field externally given, and $\boldsymbol{E}$ is an electric field induced by the magnetic field. The magnetic field is uniform in space. The circuit is placed on a flat surface whose normal unit vector is expressed by $\boldsymbol{n}$. Ignore the thickness of the conducting wire.

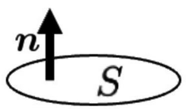


Figure 3

(Q.5) There is a square, whose side length is L, on an xy plane in a Cartesian coordinate system as depicted by broken lines in Figure 4(a). We made a closed circuit by winding a conducting wire twice in the same direction along the sides of this square. This wire has a resistance value R per the length L. Here, a magnetic field is applied uniformly in space in the z direction. The magnitude of the magnetic field B varies with time t as shown in Figure 4(b). The value of B is B_s for $t \leq t_s$, B_e for $t \geq t_e$ and varies at a constant rate for $t_s < t < t_e$. Answer the following questions for $t_s < t < t_e$. Ignore the self-inductance of the circuit and the thickness of the conducting wire.

(1) Write the electric voltage V induced in the circuit by using the variables used in the above.

(2) Describe the force on the wire, using V, R, L in about 5 lines.

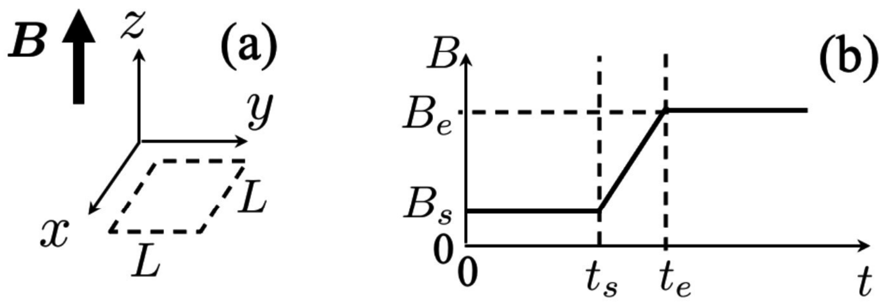


Figure 4